Last Class

10/27/16

1. A3 due next Fri
2. Update to A3 posted (bonus + hints added)

LP Relaxation for Set Cover

$$\min \sum_{S_i \in S} x_i\, c(S_i)$$

$$\text{s.t.} \sum_{S_i \text{ s.t. } j \in S_i} x_i \geq 1 \quad \forall j \in U$$

$$0 \leq x_i \underbrace{\leq 1}_{\text{redundant}}$$

Dual LP

$$\max \sum_{j \in U} y_j$$

$$\text{s.t.} \sum_{j : j \in S_i} y_j \leq c(S_i) \quad \forall S_i \in S$$

$$y_j \geq 0 \quad \forall j \in U$$

Relaxed primal C.S.:

Let $\alpha \geq 1$.

$\forall j$: either $x_j = 0$ or

$$\frac{c_j}{\alpha} \leq \sum_i a_{ij} y_i \leq c_j$$

Relaxed dual C.S.:

Let $\beta \geq 1$.

$\forall i$: either $y_i = 0$ or

$$b_i \leq \sum_{j=1} a_{ij} x_j \leq \beta b_i$$

Algorithm: Choose $\alpha = 1$, $\beta = f$, where f frequency of most common element $j \in U$ in input.

$\Rightarrow$ Primal CS for set cover LP:

$$\forall i, \quad x_i \neq 0 \Rightarrow \sum_{i \text{ s.t. } j \in S_i} y_j = c(S_i)$$

we say S_i is "tight".

$\Rightarrow$ Relaxed CS for dual:

$$\forall j, \quad y_j \neq 0 \Rightarrow 1 \leq \sum_{S_i \text{ s.t. } j \in S_i} x_i \underbrace{\leq f}_{\text{relaxation}}$$

Algorithm 15.2 (Set Cover via Primal-dual Schema) [Bar-Yehuda, Even, 1981]

(not primal feasible!)

1. Set all $x_i = 0$ and $y_j = 0$.
2. Until all elements j covered:
   Pick an uncovered element j.
   Raise y_j until some set goes "tight".
   Pick all tight sets in the cover and update $\{x_i\}$ accordingly.
   Declare all elements in the tight sets covered.
3. output set cover $\{x_i\}$.

Analysis: **Runtime:** each time run loop, we cover at least 1 new element.
$\therefore$ # iterations is poly($|U|$).

Correct: - Alg returns valid set cover since continues looping so long as $\exists$ uncovered elements.
- Note $x_i \in \{0,1\}$ $\forall i$. $\Rightarrow$ integral sol'n!
- Also dual feasible sol'n since stop increasing any y_j as soon as a set goes tight.

Approximation achieved:

Claim: Algorithm achieves f-approximation.

Pf/ By step ②, we satisfy primal C.S. condition.
Dual C.S. condition satisfied trivially by any valid set cover.

Suppose alg returns primal/dual sol'ns (x, y).

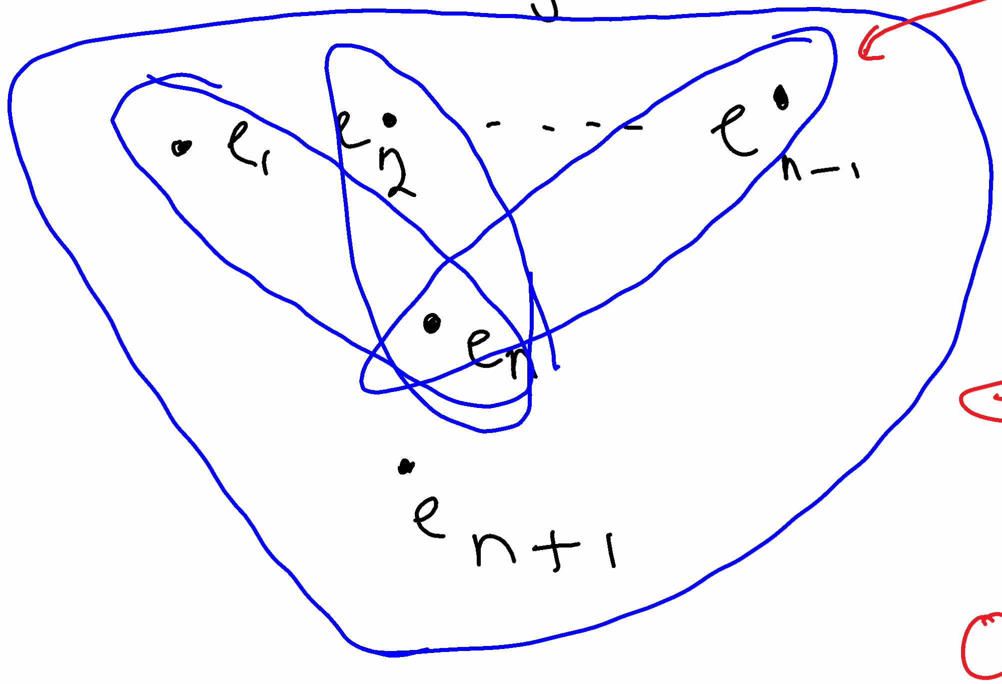
By Prop 15.1, $\sum_{S_i \in S} x_i \cdot c(S_i) \leq 1 \cdot f \cdot \sum_j y_j \leq f \cdot \sum_j y_j^* = f \cdot OPT$

$\uparrow$ primal value $\uparrow$ Prop 15.1 $\uparrow$ y_j^* dual optimal $\uparrow$ strong duality for LPs.

Q: Is our analysis tight?

A: Yes!

Consider set system:



$n-1$ 2-element sets, each cost of 1.

1 $(n+1)$-element set, cost $1+\epsilon$ for any $\epsilon > 0$.

OPT $= \boxed{1+\epsilon}$ (pick biggest set!)

Our alg: If it picks e_n is first iteration and raises y_{e_n}, and all (2)-element sets will go tight.

$\therefore$ it picks all 2-element sets.

$\therefore$ Total cost: $(n-1) \cdot 1 + (1) \cdot (1+\epsilon) = \boxed{n+\epsilon}$

since $f = n$ here, this means we saturate f-approximation ratio

$\boxed{n+\epsilon \approx f \cdot \text{OPT} = f(1+\epsilon)}$.

What about rounding-based alg for Set Cover?

Alg 14.1 (Set Cover via LP-rounding)

1. Find optimal sol'n to Set Cover LP.
2. Pick all sets S_i for which $\boxed{x_i \geq \frac{1}{f}}$ in this sol'n.

(Disadvantage: Have to use LP solver!)

Claim: Alg 14.1 is an f-approx alg.

Pf/ Correctness: Why get valid set cover?

↳ Recall from primal $\underbrace{\sum_{S_i \text{ s.t. } j \in S_i}}_{\exists \text{ at most } f \text{ terms in sum}} x_i \geq 1 \quad \forall j \in U.$

$\therefore \exists\, i$ s.t. $x_i \geq \frac{1}{f} \quad \forall j \in U$

$\therefore$ all elements of U are covered.

Why do we get approx ratio of f?

↳ In optimal sol'n to LP, any $\quad$ (obj fcn: $\sum_i x_i \cdot c(S_i)$)

$x_i^* \geq \frac{1}{f}$ is rounded to 1

$x_i^* < \frac{1}{f}$ is rounded to 0

$\therefore$ obj fcn value can increase by $\leq f$.

Let $\{x_i'\}$ be rounded sol'n.

due to this

$\therefore$ value of $\{x_i'\} \leq f \cdot (\text{value of } \{x_i^*\}) \leq f \cdot (\text{opt value for set cover}) = f \cdot OPT$.

for LP relaxation

I.P. version

∎

Chapter 16 Maximum Satisfiability

Problem 16.1 (MAX-SAT)

Input: (1) CNF formula $f : \{0,1\}^n \mapsto \{0,1\}$, ie

$$f = \bigwedge_{c \in C} c \quad \text{where each } c \text{ is OR of literals.}$$

set C of clauses

(2) non-negative weights w_c for each $c \in C$.

Output: What is max total weight of satisfied clauses over all assignments $x \in \{0,1\}^n$.

e.g. $f = (\underbrace{x_1 \vee x_2 \vee x_3}_{c_1}) \wedge (\underbrace{\bar{x}_1 \vee x_3 \vee \bar{x}_5 \vee x_6}_{c_2}) \wedge (\underbrace{\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3}_{c_3})$

$w_{c_1} = 2, \; w_{c_2} = 3, \; w_{c_3} = 9.$

$OPT = 2 + 3 + 9 = 14.$

NP-complete!

Each x_i is a variable.

Each x_i or $\bar{x}_i$ is a literal.

Notation: Random variable W - total weight of satisfied clauses
W_c - weight contributed to W by clause c.

$$\therefore \quad W = \sum_{c \in C} W_c .$$

linearity of expectation

$$E[W] = E\Big[\sum_{c \in C} W_c\Big] = \sum_{c \in C} E[W_c] = \sum_{c \in C} w_c \cdot \Pr[\text{satisfying clause } c]$$

Alg strategy: Design 2 algs, and combine them:

1. Random assignment ← good for large clauses
2. LP relaxation + randomized rounding

↑ good for small clauses

Alg 1: Random Assignment (for large clauses)

1. Set each $x_i = 1$ w.p. $\frac{1}{2}$ independently & uniformly at random.

How well does this do? ie what is $E[W]$?

$\boxed{\text{Lemma 16.2}}$ If $c \in C$ has k literals, then $E[W_c] = \left(1 - \frac{1}{2^k}\right) \cdot w_c$.

Pf | $E[W_c] = Pr[\text{satisfy } c] \cdot w_c$

eg $c = (x_1 \vee \bar{x}_2 \vee \cdots x_k)$

$$Pr(\text{satisfy } c] = 1 - Pr[\text{fail to satisfy } c]$$
$$= 1 - \frac{1}{2^k}$$

$\Leftarrow$ only 1 assignment out of all 2^k possible ones fails!

$\therefore$ For whole alg:

Lem. 16.2

size of clause c

$$E[W] = \sum_{c \in C} E[W_c] = \sum_{c \in C} \left(1 - \frac{1}{2^{k_c}}\right) w_c \geq \left(1 - \frac{1}{2^{k_{min}}}\right) \sum_{c \in C} w_c \geq \left(1 - \frac{1}{2^{k_{min}}}\right) OPT.$$

k_{min} is smallest clause size

$\sum_{c \in C} w_c$ means satisfy all clauses.